Plasticity and Fracture in Porous Media: Uniqueness, Mesh Dependency, and Fluid Transport

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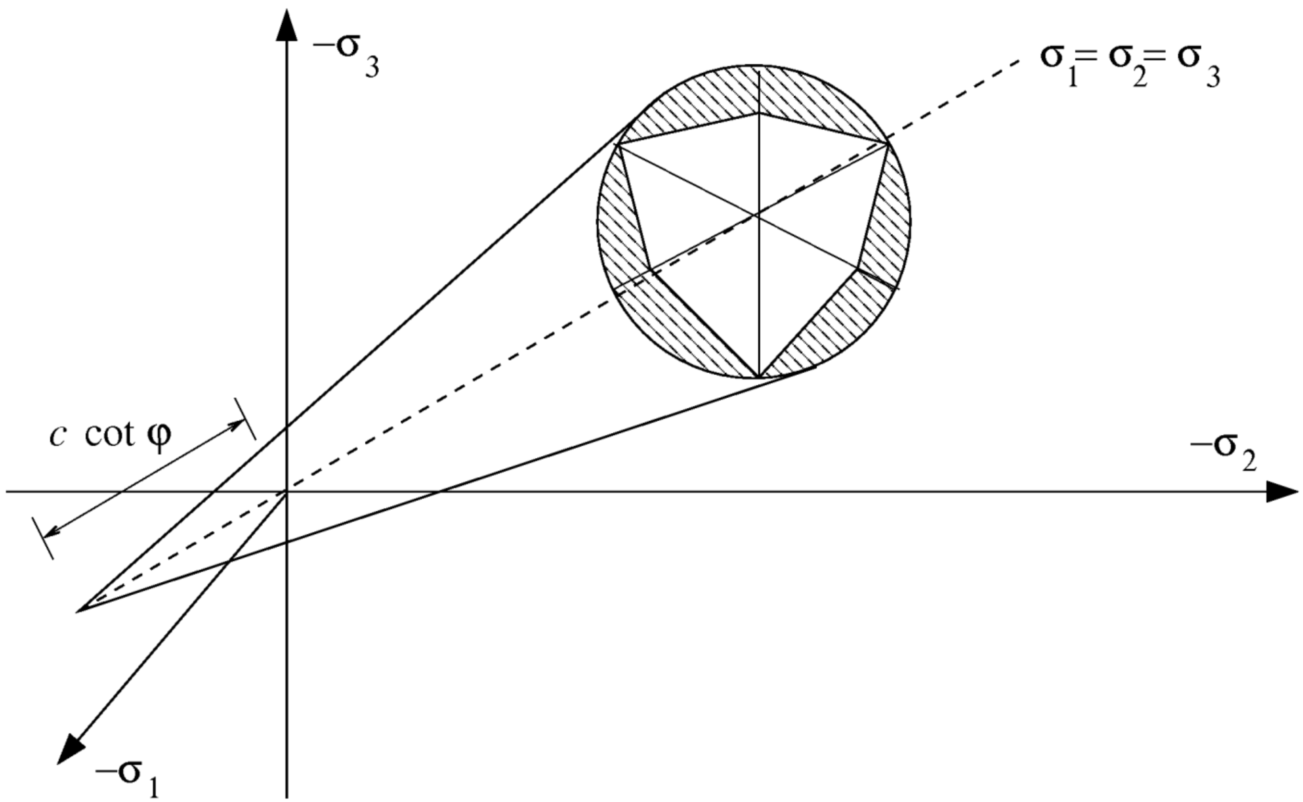
Mechanical behavior of geomaterials (rocks, soils):

- Non-linearity from the onset of loading
- Strong pressure dependence
- Coupling between shear loading and volumetric response
- Time dependence
- Coupling with diffusion-like phenomena: fluids, thermal heating

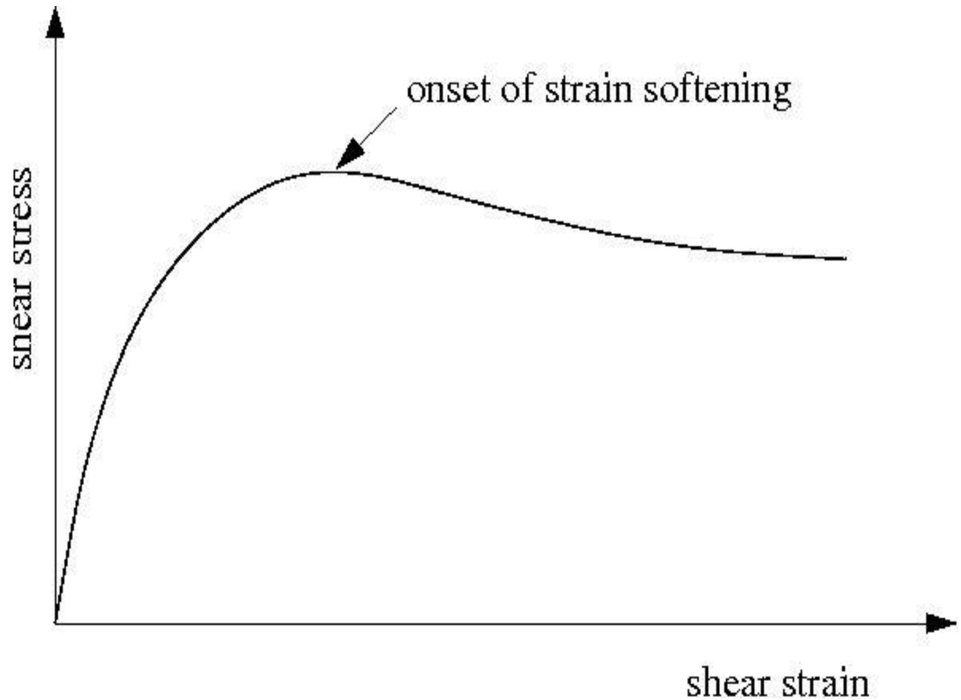
Approaches to constitutive modelling:

- Plasticity
  - pressure dependence: Mohr-Coulomb, Drucker-Prager, …
  - strain softening
  - non-associated plastic flow
- Fracture modelling:
  - Linear Elastic Fracture Mechanics (LEFM)
  - Non-linear fracture models: cohesive-zone approach
- Damage models
  - scalar valued
  - tensor valued

Pressure dependence: Mohr-Coulomb, Drucker-Prager, …



Strain softening

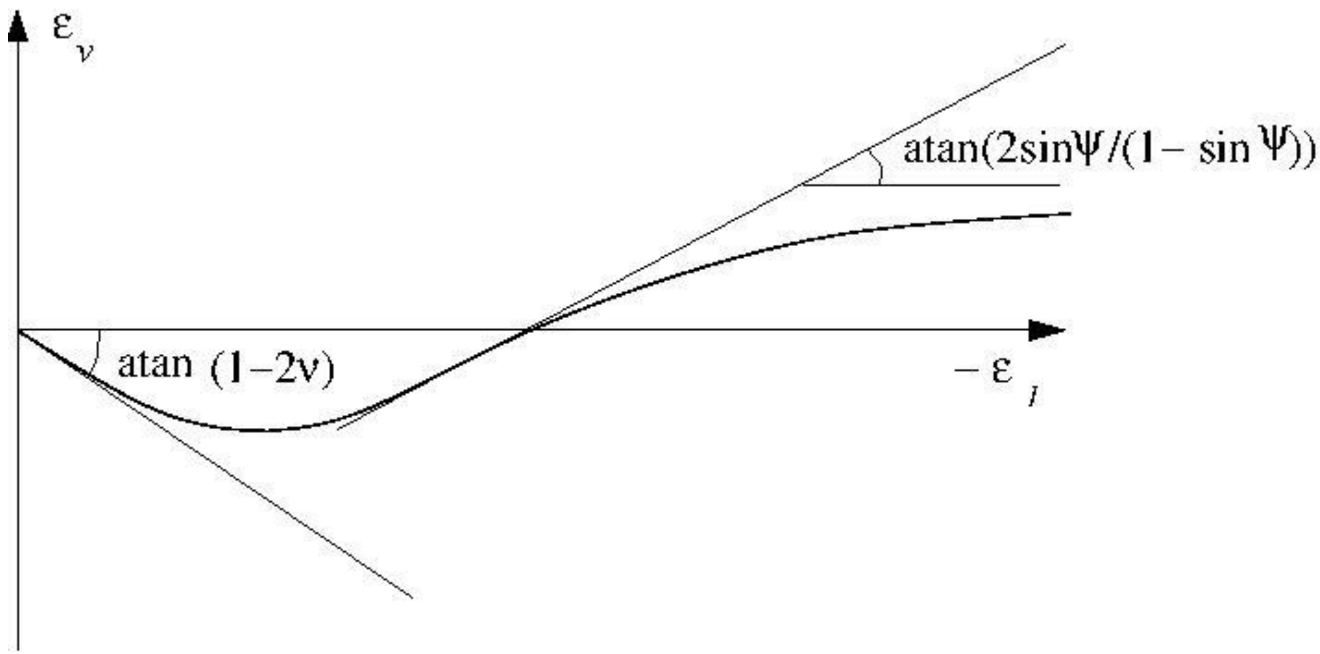


Consequence:

$$\dot{\sigma} : \dot{\epsilon} < 0$$

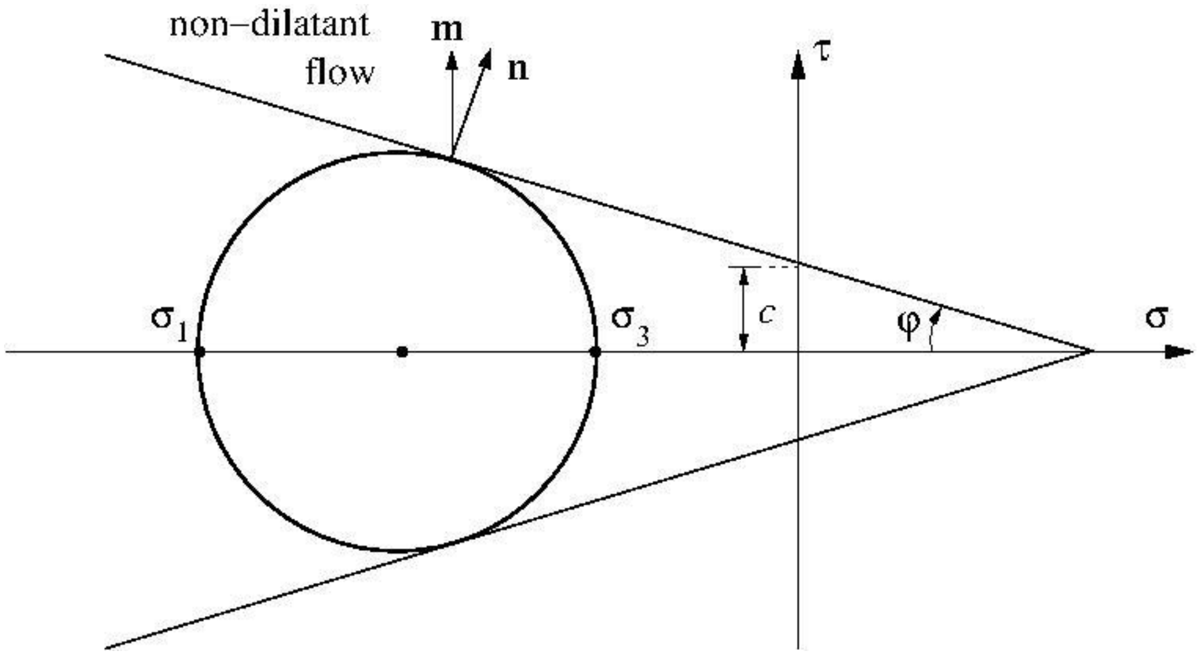
Loss of material stability!

Non-associated plastic flow



Typical experimental outcome

Non-associated plastic flow



Elasto-plastic stiffness:

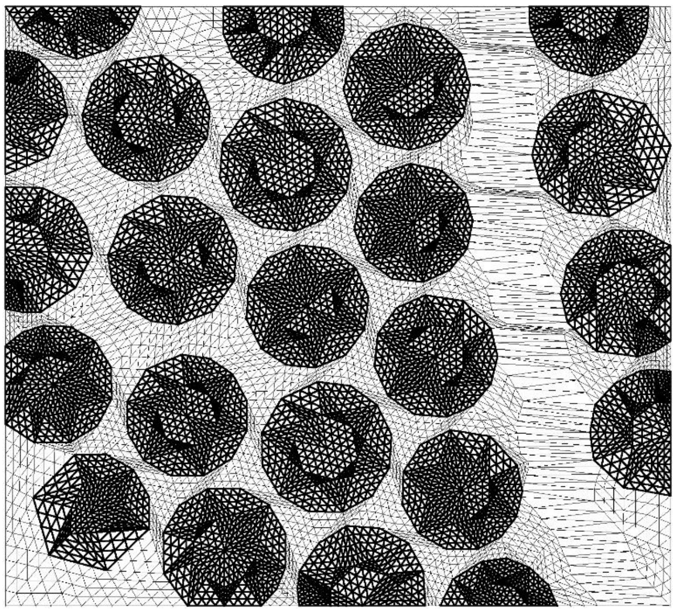
$$\mathbf{D} = \mathbf{D}^{\mathrm{e}} - \frac{\mathbf{D}^{\mathrm{e}}\mathbf{m}\mathbf{n}^{\mathrm{T}}\mathbf{D}^{\mathrm{e}}}{h + \mathbf{n}^{\mathrm{T}}\mathbf{D}^{\mathrm{e}}\bar{\mathbf{m}}}$$

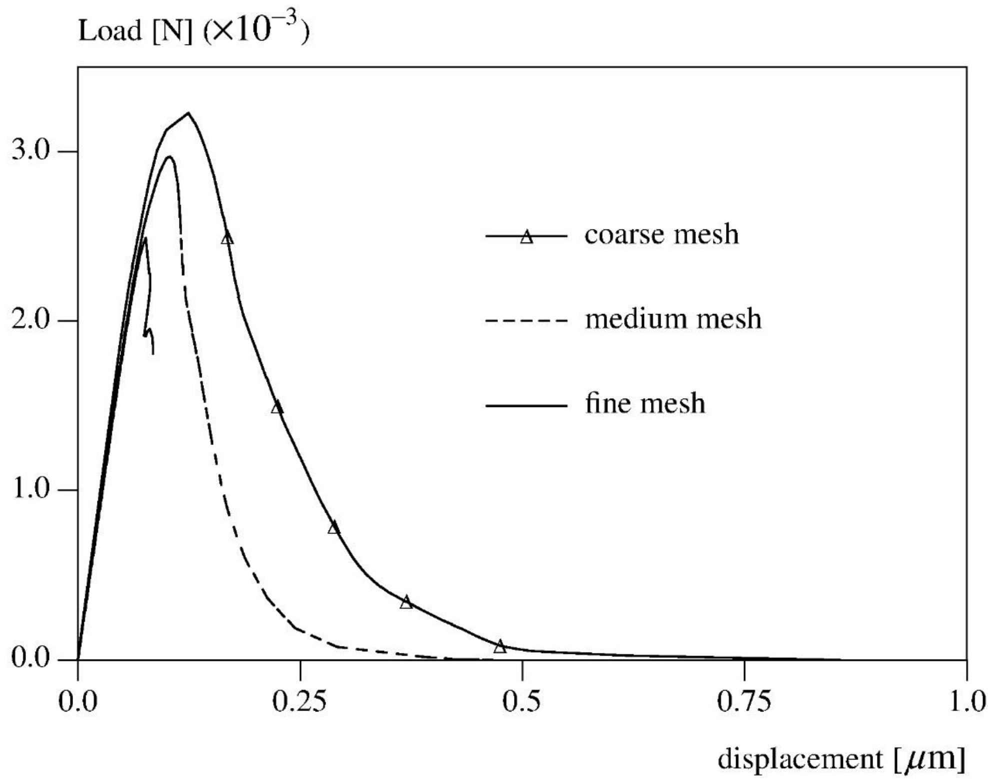
Non-symmetry!

Again: possible **loss of material stability:** $\dot{\boldsymbol{\sigma}} : \dot{\boldsymbol{\epsilon}}^p < 0$

Negligible or no elastic strain rates: $\dot{\boldsymbol{\sigma}} : \dot{\boldsymbol{\epsilon}} < 0$

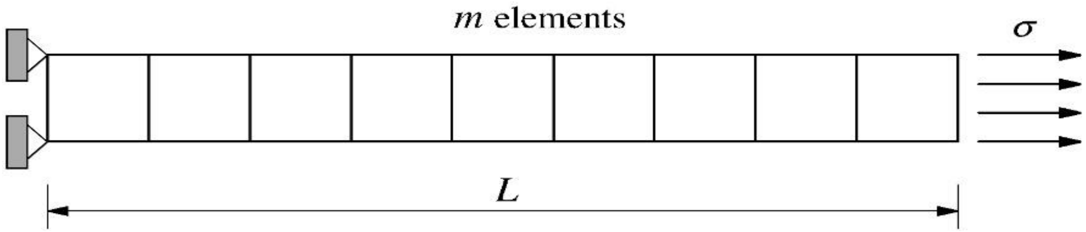
Example of **mesh dependence** and **lack of convergence** upon mesh refinement



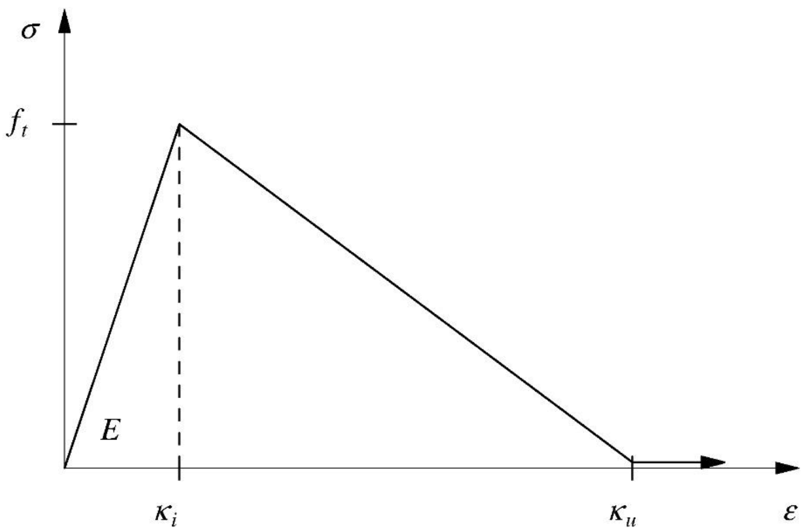


Fibre-reinforced epoxy layer (plane-strain conditions, horizontal loading)

1D example of mesh dependence



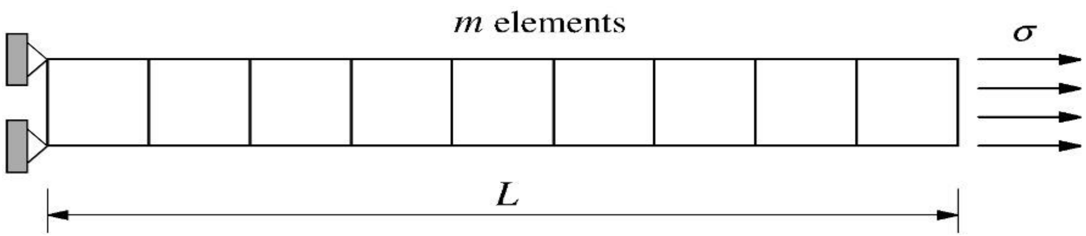
Input: strain-softening relation



$$\sigma = E\epsilon$$

$$\sigma = f_t + h(\epsilon - \kappa_i)$$

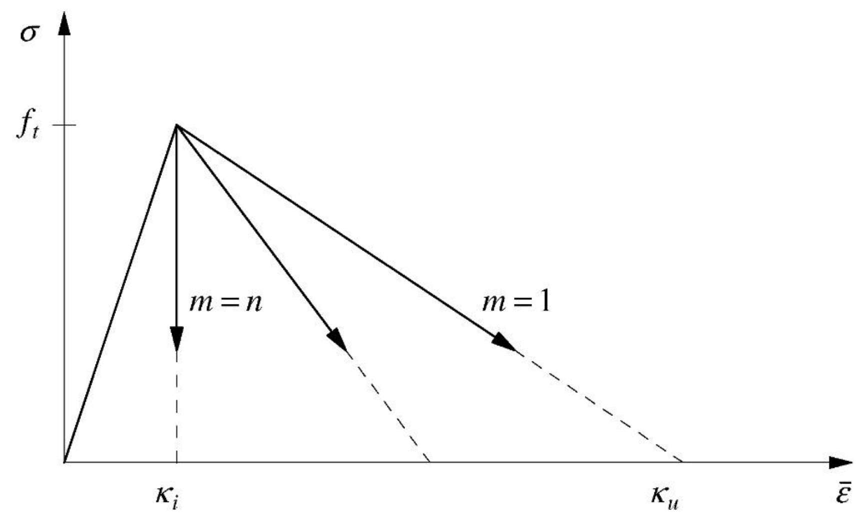
1D example of mesh dependence



$$\bar{\epsilon} = \frac{\sigma}{E} + \frac{n(\sigma - f_t)}{mE}$$

- Energy dissipation dependent on discretisation
- Zero energy dissipation when *m* goes to infinity

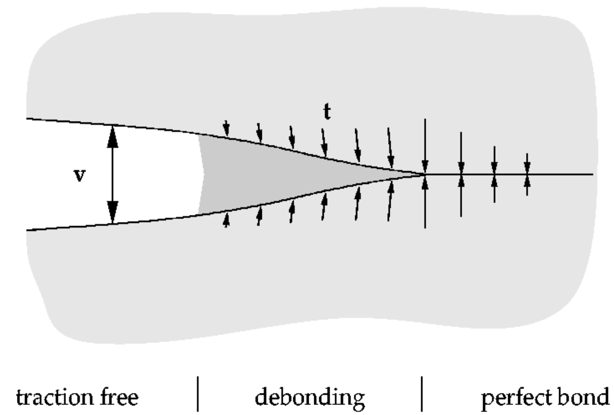
Output: structural response of bar



The cohesive approach to fracture

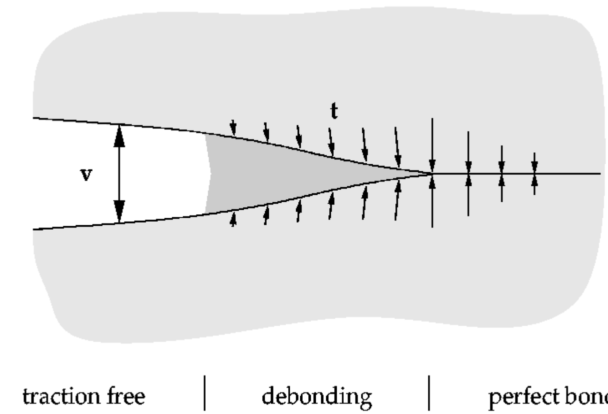
(Dugdale – Barenblatt approach)

- There exists a finite zone ahead of the crack tip in which fibre breakage, micro-cracking and plasticity occur

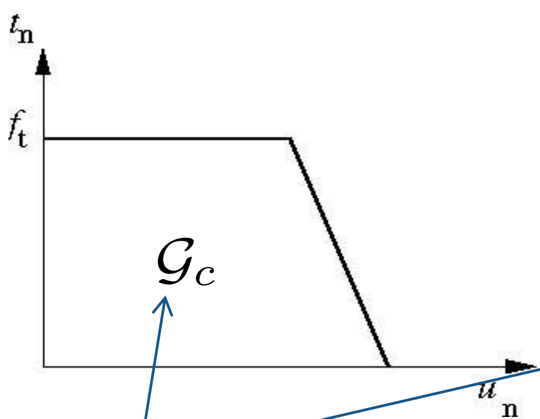


- Important parameters:
  - Tensile strength
  - Fracture energy: *Energy required to create a unit area of crack*
  - Shape of decohesion curve (important for quasi-brittle failure)

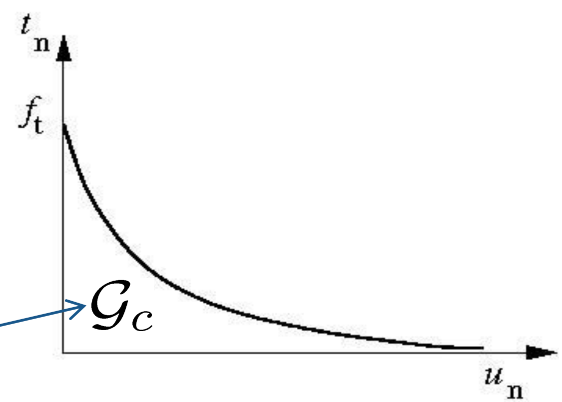
- Shape of decohesion curve (continued…)



Ductile failure
(metals)



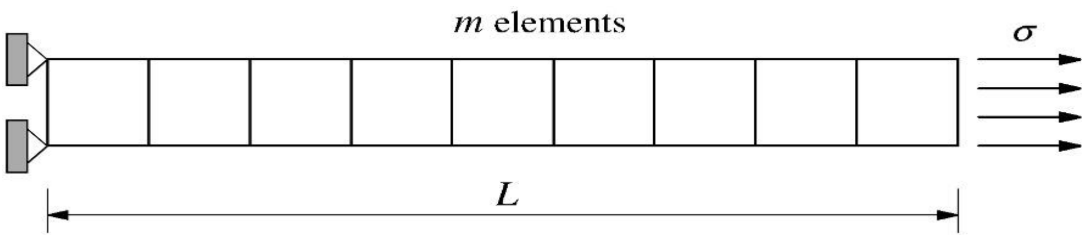
Quasi-brittle failure
(concrete, rocks)



$$\boxed{\mathcal{G}_c = \int t_n \, \mathrm{d}v_n \Rightarrow t_n = \frac{\partial \mathcal{G}_c}{\partial v_n} \Rightarrow t_n = t_n(v_n, \kappa)}$$

$\longrightarrow$ Discrete traction separation law

1D example of mesh dependence



Smearing out of the fracture energy over a finite width:

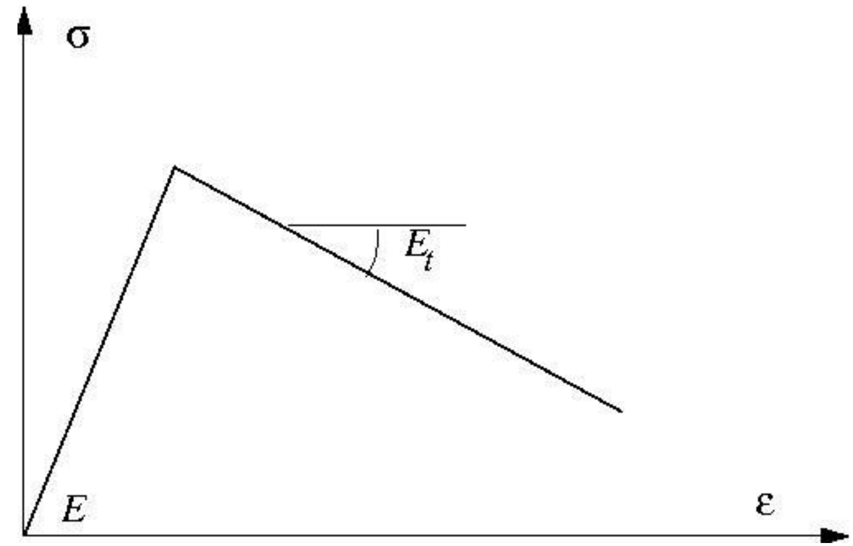
$$\mathcal{G}_c = \int_{n=0}^{w} \int_{\epsilon=0}^{\infty} \sigma \mathrm{d}\epsilon(n) \mathrm{d}n$$

Average strain independent of discretisation & **size effect** introduced:

$$\bar{\epsilon} = \frac{\sigma}{E} - \frac{2\mathcal{G}_c(\sigma - f_t)}{L f_t^2}$$

But, it is not the full story!

One-dimensional wave equation:



$$E_t \frac{\partial^2 u}{\partial x^2} = \rho \frac{\partial^2 u}{\partial t^2}$$

Phase velocity:

$$c_f = \sqrt{\frac{E_t}{\rho}} \longrightarrow$$

Imaginary wave speeds

- Consequences of strain softening and non-associated plasticity:
  - Loss of material stability (Diderot / d'Alembert: stability = "rigid, unmovable")

$$\dot{\boldsymbol{\sigma}} : \dot{\boldsymbol{\epsilon}} < 0 \quad \longleftarrow$$

Only equivalent to Lyapunov's definition
for small-strain elasticity

$$\dot{\boldsymbol{\sigma}} = \mathbf{D} : \dot{\boldsymbol{\epsilon}} \quad \longrightarrow \quad \dot{\boldsymbol{\epsilon}} : \mathbf{D} : \dot{\boldsymbol{\epsilon}} < 0 \quad \longrightarrow \quad \det(\mathbf{D}^{\mathrm{sym}}) = 0$$

  - Loss of structural stability

$$\int_V \dot{\boldsymbol{\sigma}} : \dot{\boldsymbol{\epsilon}} \mathrm{d}V < 0 \quad \longrightarrow \quad \dot{\mathbf{a}}^{\mathrm{T}} \left(\int_V \mathbf{B}^{\mathrm{T}} \mathbf{D} \mathbf{B} \mathrm{d}V \right) \dot{\mathbf{a}} < 0$$

$$\mathbf{K} = \int_V \mathbf{B}^{\mathrm{T}} \mathbf{D} \mathbf{B} \mathrm{d}V \quad \longrightarrow \quad \det(\mathbf{K}^{\mathrm{sym}}) = 0$$

- Consequences of strain softening and non-associated plasticity:
  - Loss of uniqueness

$$\mathbf{K}\dot{\mathbf{a}} = \dot{\lambda}\mathbf{f}^{\text{ext}} \quad \longrightarrow \quad \mathbf{K}\Delta\dot{\mathbf{a}} = \mathbf{0} \quad \rightarrow \quad \det(\mathbf{K}) = 0$$

  - Loss of ellipticity

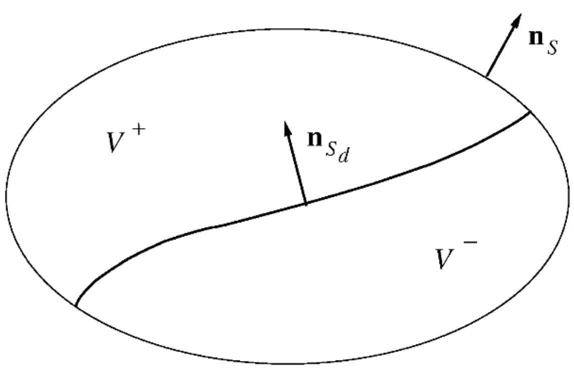
$$\mathbf{u} = \bar{\mathbf{u}} + \mathcal{H}_{S_d}\tilde{\mathbf{u}} \quad \longrightarrow \quad \boldsymbol{\epsilon} = \nabla^{\text{sym}}\bar{\mathbf{u}} + \mathcal{H}_{S_d}\nabla^{\text{sym}}\tilde{\mathbf{u}} + \delta_{S_d}(\tilde{\mathbf{u}} \otimes \mathbf{n}_{S_d})^{\text{sym}}$$

$$[\![\dot{\mathbf{t}}_d]\!] = \mathbf{n}_{S_d} \cdot [\![\dot{\boldsymbol{\sigma}}]\!] \quad \longrightarrow \quad [\![\dot{\mathbf{t}}_d]\!] = \mathbf{n}_{S_d} \cdot \mathbf{D} : [\![\dot{\boldsymbol{\epsilon}}]\!]$$

$$[\![\dot{\mathbf{t}}_d]\!] = \zeta(\mathbf{n}_{S_d} \cdot \mathbf{D} \cdot \mathbf{n}_{S_d}) \cdot \dot{\tilde{\mathbf{u}}}$$

$$\det(\mathbf{n}_{S_d} \cdot \mathbf{D} \cdot \mathbf{n}_{S_d}) = 0$$

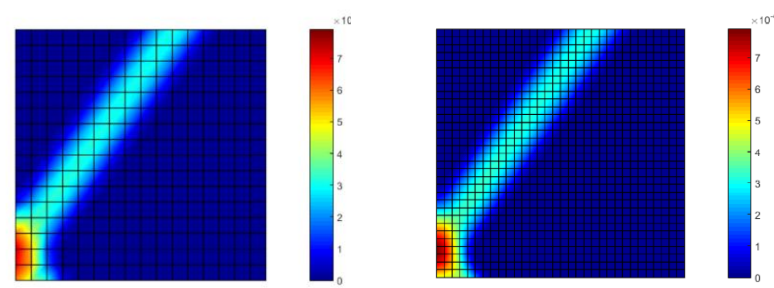
Mesh sensitivity!



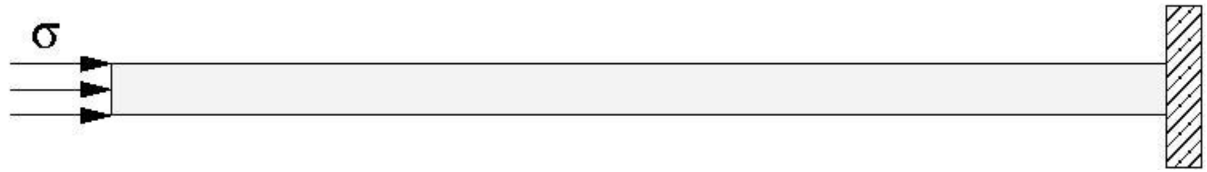
Loss of ellipticity (or hyperbolicity in dynamics) is the underlying cause of **mesh dependence.** It is a fundamental mathematical issue, and occurs for **any numerical method.**

Possible solutions:

- Include viscosity / rate dependence
- Introduce non-local terms through upscaling / homogenisation
- Include more “physics” in the constitutive description, e.g.,
  - Thermal heating
  - Fluid flow, however …

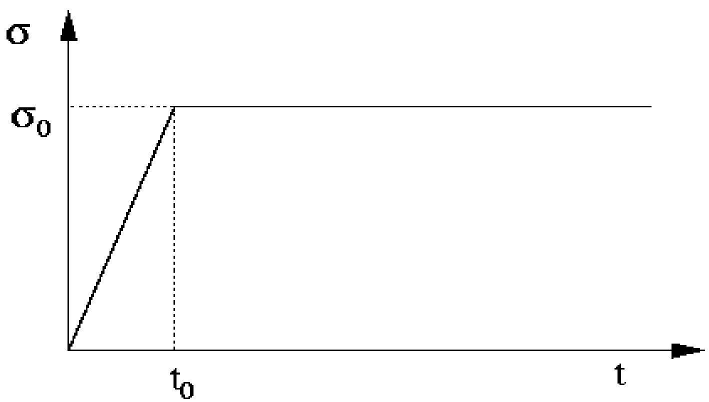


Impact loading of 1D fluid-saturated medium

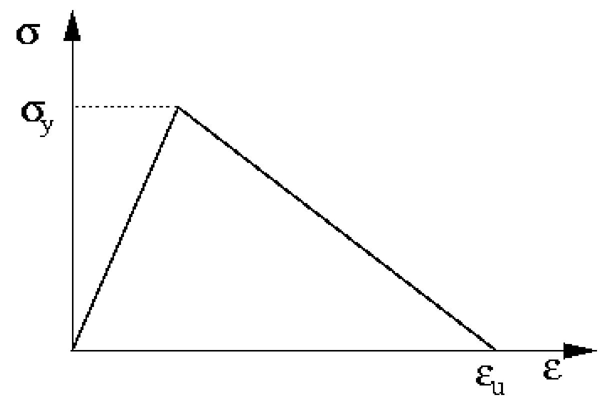


- Finite differences in space
- Fully explicit time integration

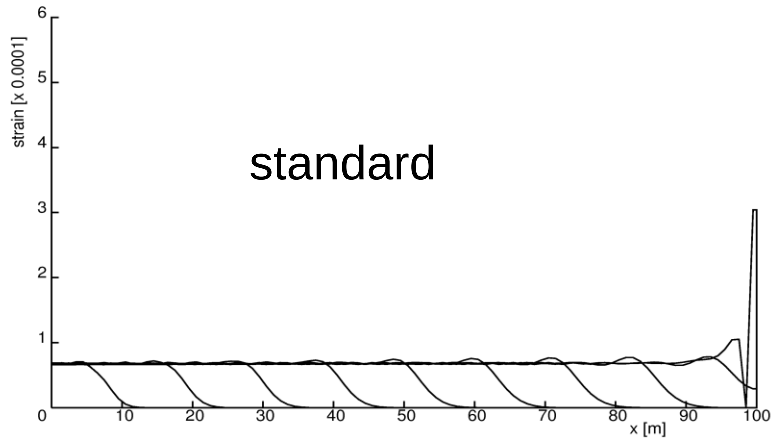
→ to avoid numerical regularisation

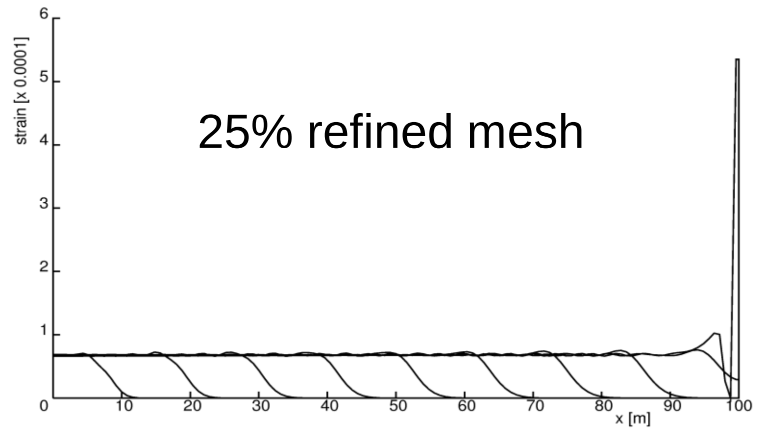


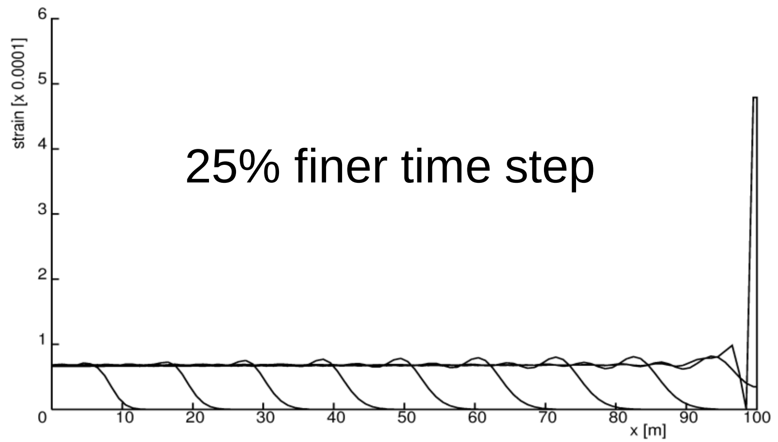
loading scheme



softening relation







Solution depends on:

- Grid spacing
- Time step

The World is neither continuous, nor discontinuous

The perception of continuity, or of a discontinuity, depends on the level of observation

At a certain level of observation a discontinuity can be *modelled* as a continuity (smeared concept), but equally well as a discontinuity (discrete concept)

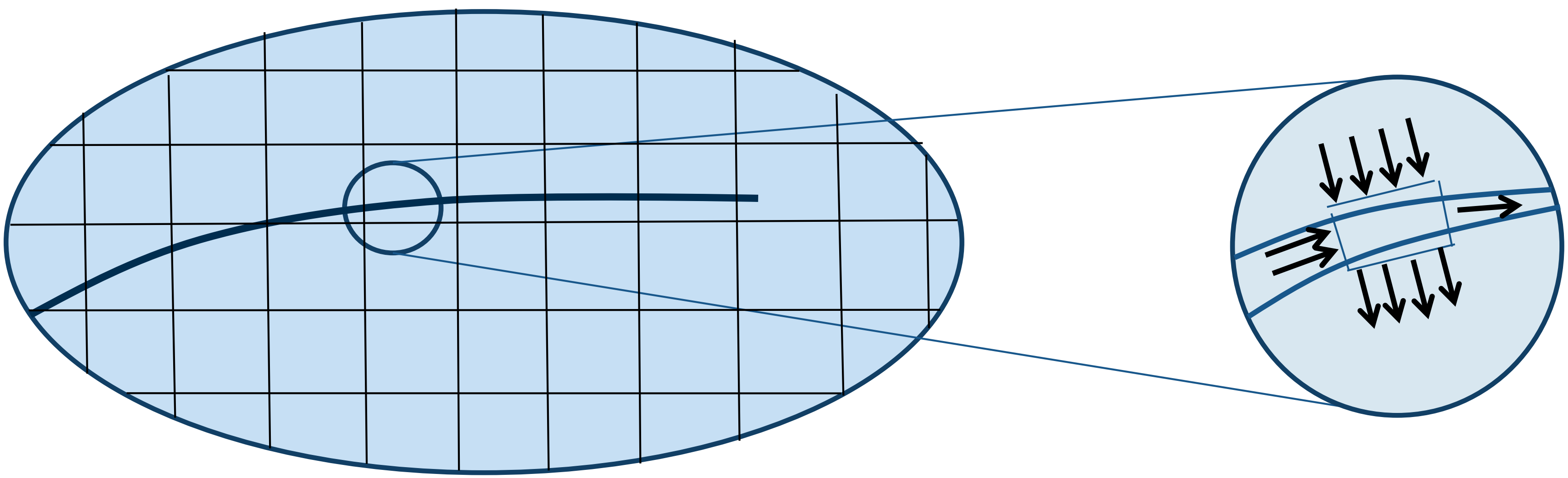
In some cases it is meaningful to use a discrete concept

We wish to model the propagation of cracks or shear bands:

- In a porous medium
- Including the possibility to transport fluid in the cracks
- For relatively large domains
- Independent of the underlying discretisation
- Which can be extended to include gas phases etc.

Multi-scale approach:

Crack propagation not biased by finite element discretisation



Mass and momentum balance on a subgrid scale

Structured or unstructured finite element discretisation

Multi-phase medium:

Balance of momentum

$$\nabla \cdot \boldsymbol{\sigma}_\pi + \widehat{\mathbf{p}}_\pi = \rho_\pi \frac{\partial \mathbf{v}_\pi}{\partial t}$$

$$\sum_{\pi=s,f} \widehat{\mathbf{p}}_\pi = \mathbf{0}$$

Balance of mass

$$\frac{\partial \rho_\pi}{\partial t} + \rho_\pi \nabla \cdot \mathbf{v}_\pi = 0$$

Balance of momentum for mixture:

$$\nabla \cdot \boldsymbol{\sigma} = \rho \dot{\mathbf{v}}_s$$

Balance of mass for mixture:

$$\alpha \nabla \cdot \mathbf{v}_s + n_f \nabla \cdot (\mathbf{v}_f - \mathbf{v}_s) + Q^{-1} \frac{\partial p}{\partial t} = 0$$

Balance of momentum for mixture (weak form):

$$\int \rho\, \boldsymbol{\eta} \cdot \dot{\mathbf{v}}_s \ \mathrm{d}\Omega + \int_\Omega (\nabla \cdot \boldsymbol{\eta}) \cdot \boldsymbol{\sigma} \ \mathrm{d}\Omega +$$

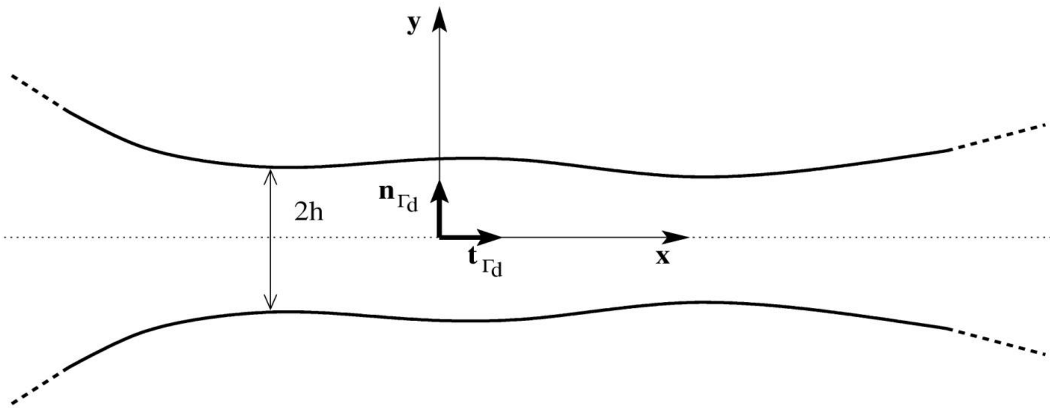
$$\underline{\int_{\Gamma_d} [\![\boldsymbol{\eta} \ \cdot \boldsymbol{\sigma}]\!] \cdot \mathbf{n}_{\Gamma_d} \ \mathrm{d}\Gamma} = \int_\Gamma \boldsymbol{\eta} \cdot \mathbf{t}_p \ \mathrm{d}\Omega$$

Balance of mass for mixture (weak form):

$$- \int_\Omega \alpha \zeta \nabla \cdot \mathbf{v}_s \ \mathrm{d}\Omega + \int_\Omega k_f \nabla \zeta \cdot \nabla p \ \mathrm{d}\Omega$$

$$- \int_\Omega \zeta Q^{-1} \dot{p} \ \mathrm{d}\Omega + \underline{\int_{\Gamma_d} \mathbf{n}_{\Gamma_d} \cdot [\![\zeta \ n_f (\mathbf{v}_f - \mathbf{v}_s)]\!] \mathrm{d}\Gamma}$$

$$= \int_\Gamma \zeta \mathbf{n}_\Gamma \cdot \mathbf{q}_p \ \mathrm{d}\Gamma$$



Subscale model for flow inside crack or shear zone:

Mass balance in cavity: $\dot{\rho}_f + \rho_f \boxed{\nabla \cdot \mathbf{v} = 0}$

Jump in normal fluid velocity: $[\![w_f]\!] = -\int_{-h}^{h} \frac{\partial v}{\partial x} \mathrm{d}y$

Momentum balance: $v(y) = \frac{1}{2\mu}\frac{\partial p}{\partial x}(y^2 - h^2) + v_f$

Continuity for $y = \pm h$:

$$v_f = (\mathbf{v}_s - {n_f}^{-1} k_f \nabla p) \cdot \mathbf{t}_{\Gamma_d}$$

Coupling term (cf Reynolds' equation):

$$n_f [\![w_f - w_s]\!] =$$

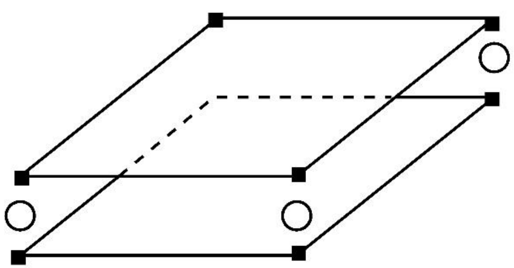
$$n_f \left(\frac{2h^3}{3\mu} \frac{\partial^2 p}{\partial x^2} + \frac{2h^2}{\mu} \frac{\partial p}{\partial x} \frac{\partial h}{\partial x} - 2h \frac{\partial v_f}{\partial x} - 2 \frac{\partial h}{\partial t} \right)$$

Consequences:

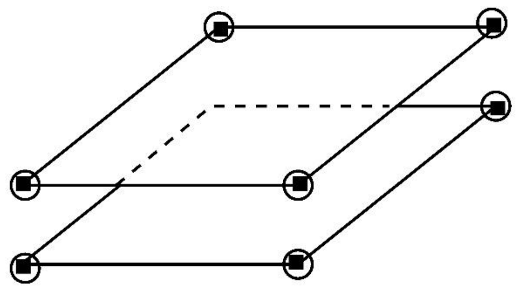
- non-linearity
- higher-order interpolation for p
- non-symmetry

Discretisation: assumptions for discontinuity:

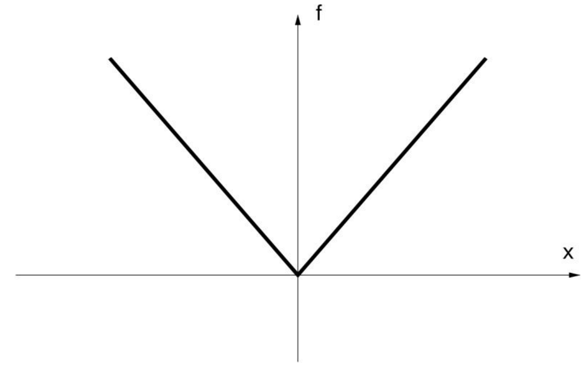
- For solid: $\mathbf{u} = \bar{\mathbf{u}} + \mathcal{H}_{\Gamma_d}\tilde{\mathbf{u}}$
- For fluid?

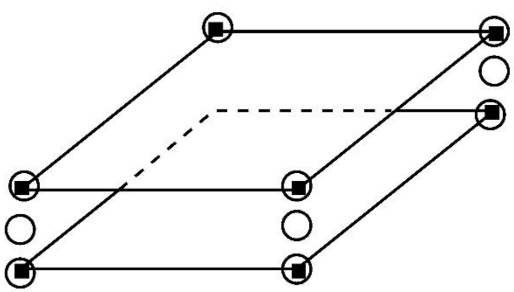


(a)



(b)

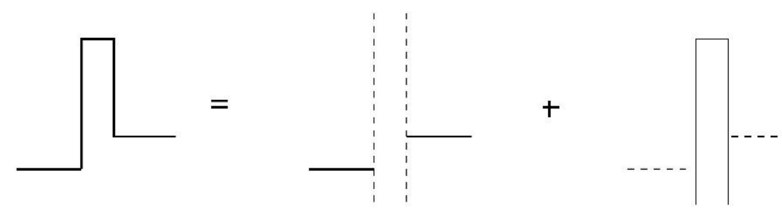




(c)

■ displacement degrees of freedom

○ pressure degree of freedom



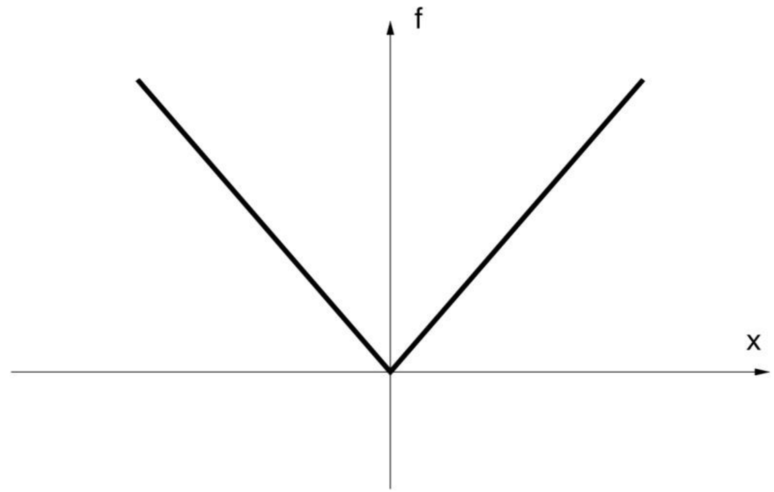
Example for single pressure degree of freedom:

- For solid: $\mathbf{u} = \bar{\mathbf{u}} + \mathcal{H}_{\Gamma_d}\tilde{\mathbf{u}}$
- For fluid: $p = \bar{p} + \mathcal{D}_{\Gamma_d}\tilde{p} \longrightarrow \nabla p = \nabla\bar{p} + \mathcal{H}_{\Gamma_d}\nabla\tilde{p}$

discontinuity in pressure gradient

$$\equiv$$

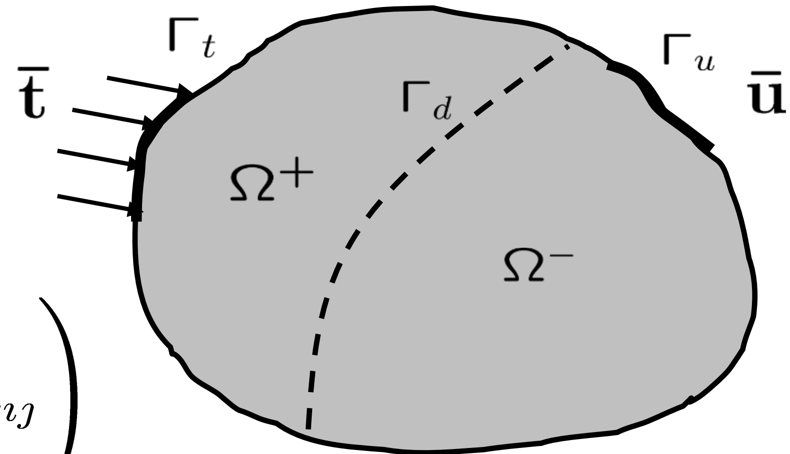
discontinuity in normal velocity



- **Discontinuity also there for other pressure discretisations**

Discretisation via
enhanced interpolation:

$$u(\mathbf{x}) = \sum_{\imath=1}^{n} \phi_\imath(\mathbf{x}) \left(\bar{a}_\imath + \sum_{\jmath=1}^{m} \psi_\jmath(\mathbf{x}) \tilde{a}_{\imath\jmath} \right)$$



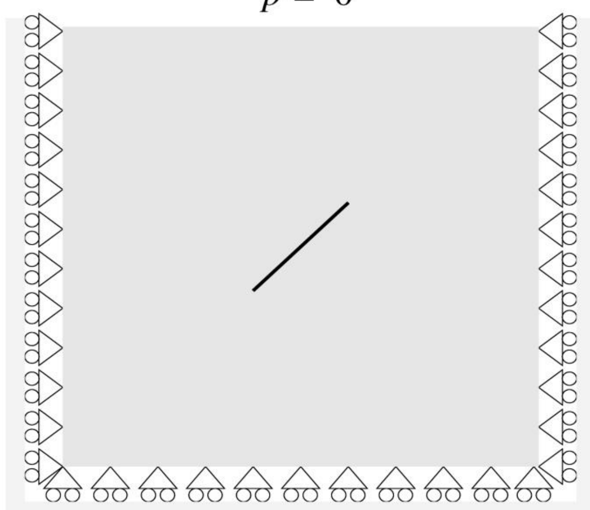
This decouples crack path from initial discretisation:
Cracks can run through finite elements!

Stationary cracks in a fluid-saturated medium

Linear-elastic fracture mechanics

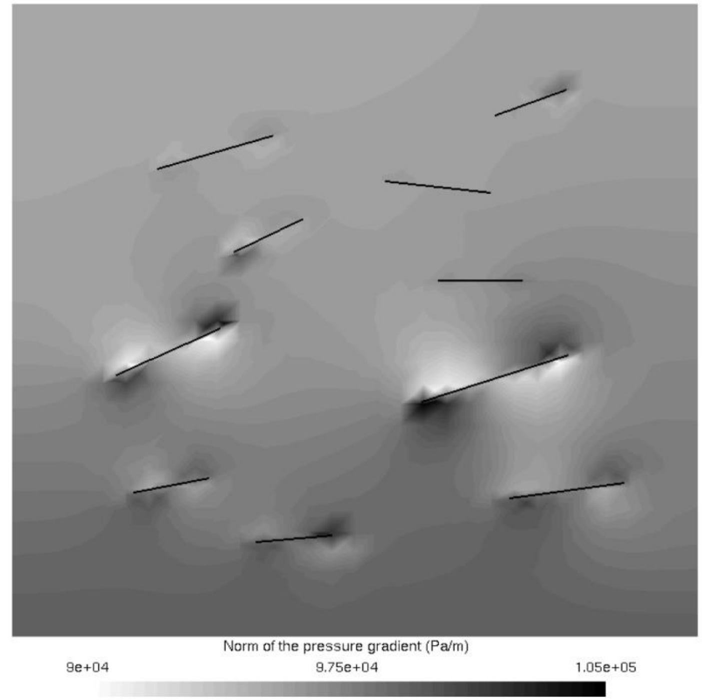
$p = 0$

$\mathbf{q} \cdot \mathbf{n} = 0$



$\mathbf{q} \cdot \mathbf{n} = 0$

$\mathbf{q} \cdot \mathbf{n} = -q_o$



Pressure gradient

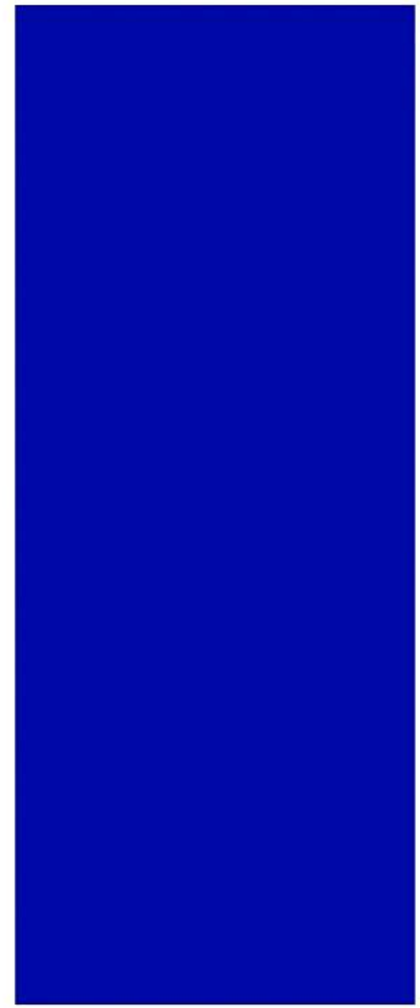
Structured mesh (40 x 40)

Quasi-static loading

Biaxial test with
an initial imperfection
under axial compression

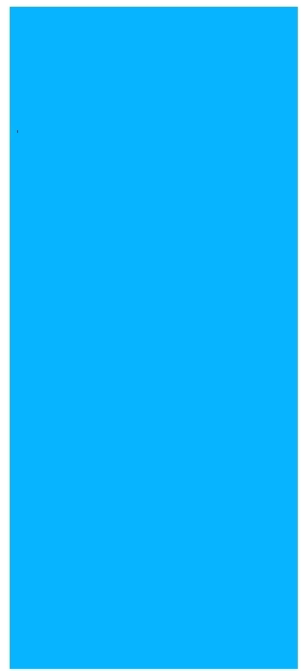
Tresca-like criterion for
inception of shear band

pressure

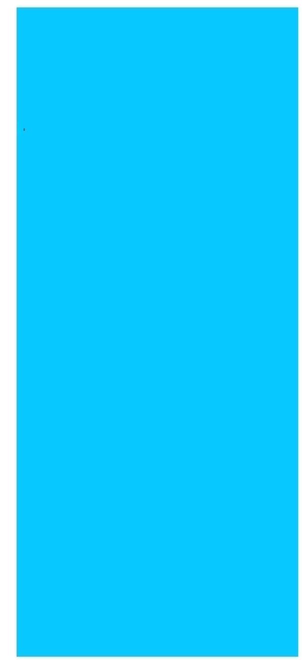


Extension to dynamic loading and Coulomb criterion for initiation

displacement

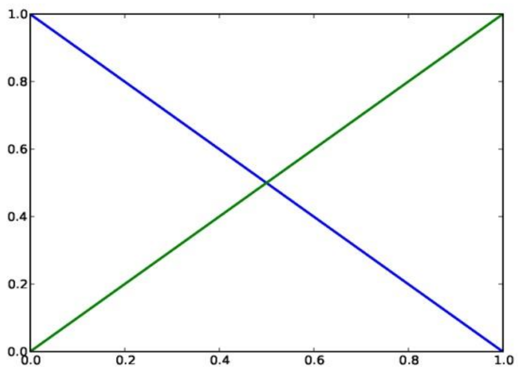


pressure

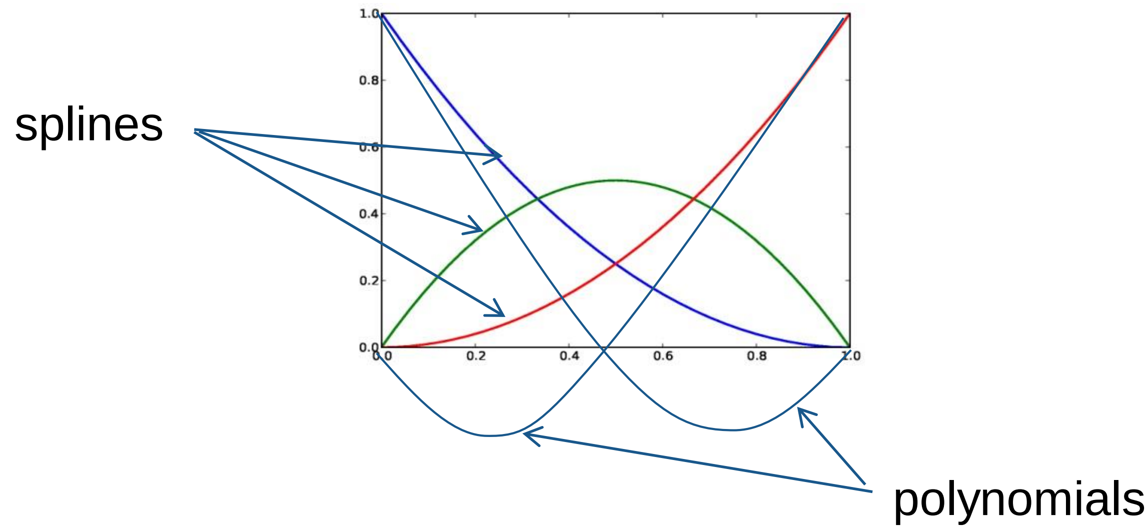


high ductility

Isogeometric analysis: use spline functions for interpolation instead of Lagrange polynomials

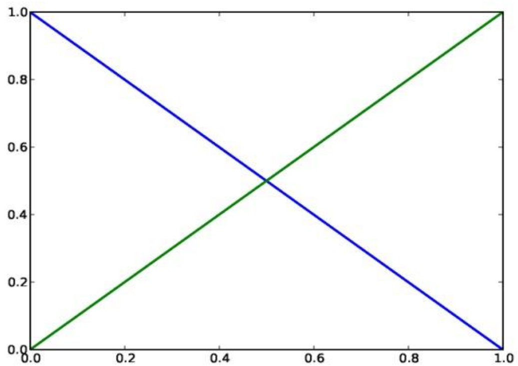


Linear basis functions:
no difference between
splines and polynomials



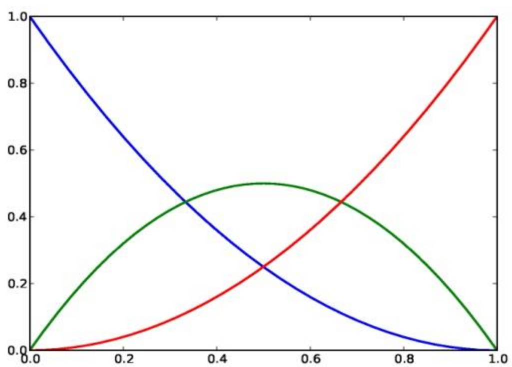
Quadratic basis functions

➢ Discrete fracture using *isogeometric analysis*

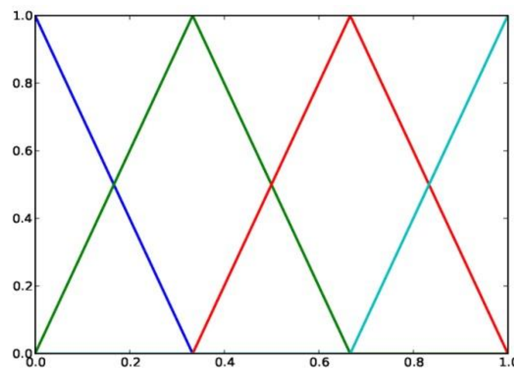


More accurate stress calculation

B-spline basis functions for $p = 1$

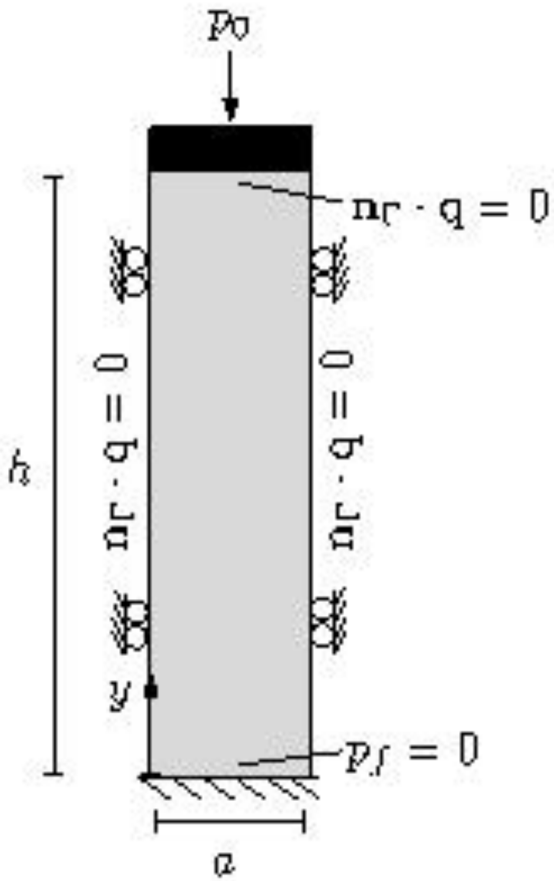


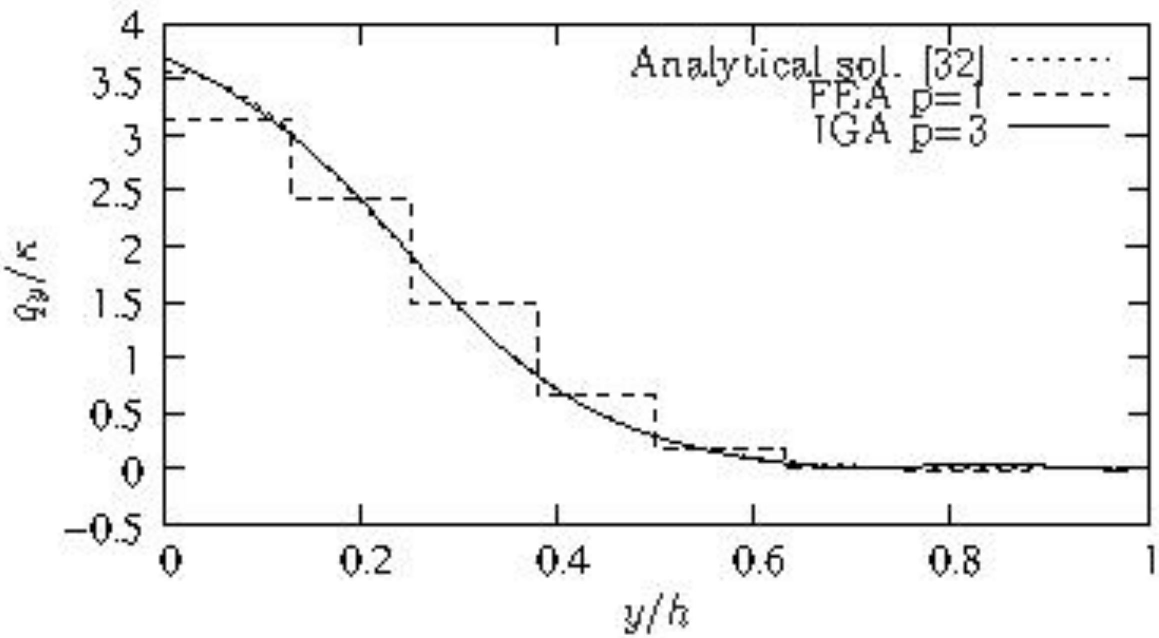
order elevation



knot insertion
=
lowering continuity

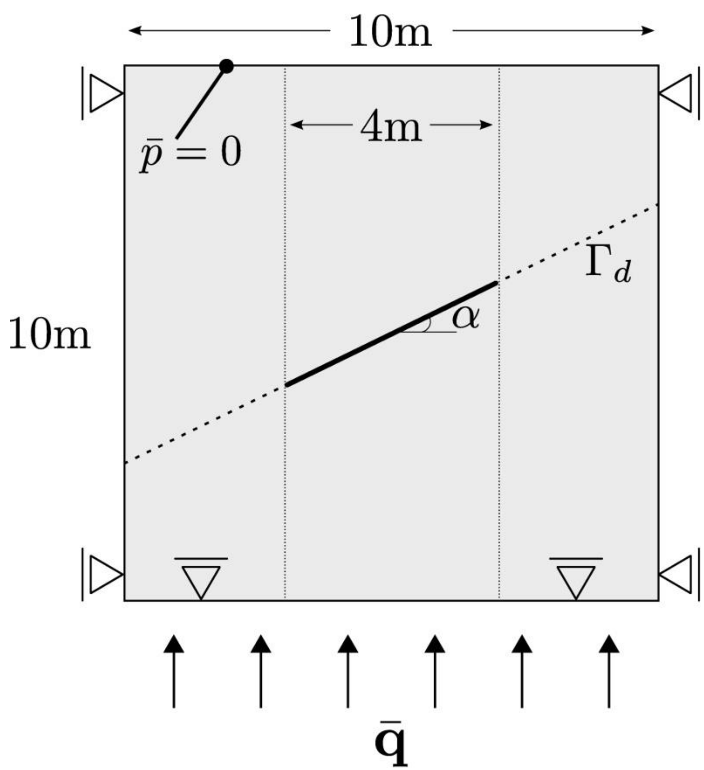
Similarly, better flow calculation in porous media

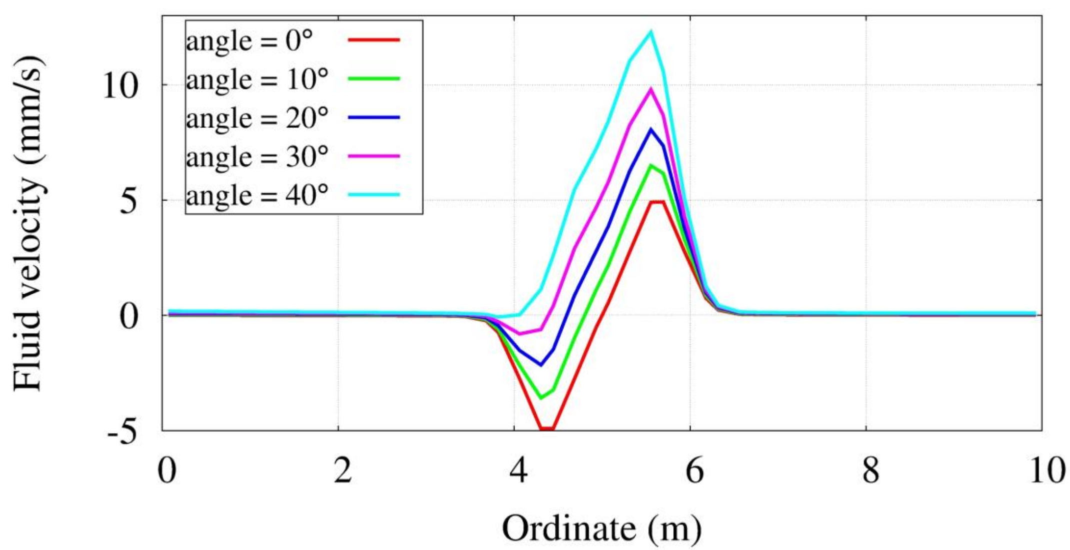




automatic satisfaction of local mass balance

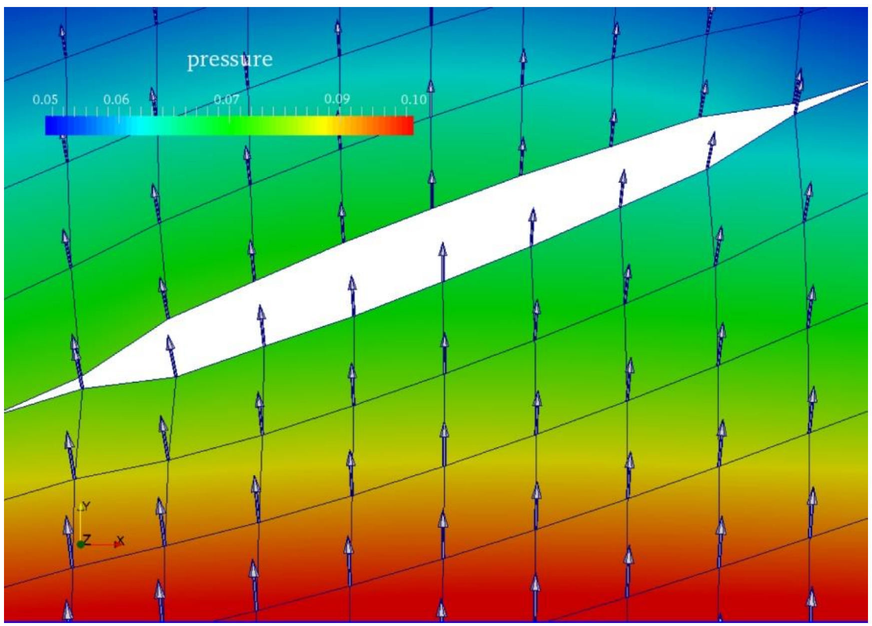
Parameter study: The influence of presence and direction of cracks on the flow pattern



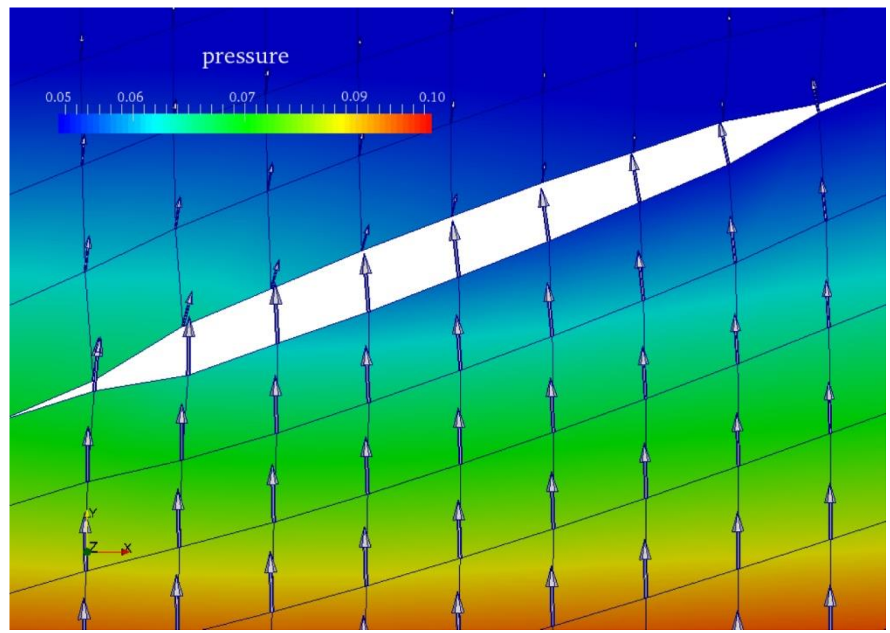


Fluid velocity in cracks is several orders of magnitude higher than that in porous medium

Results in terms of pressures and flow patterns



without micro-flow



with micro-flow

- Computations that involve strain softening and/or non-associated plastic flow show mesh dependence. This is caused by loss of ellipticity, and occurs for **any** type of discretisation method.
- Solutions involve an enhancement of the constitutive model, e.g. through addition of viscosity, non-local terms, or through inclusion of diffusion phenomena.
- Subgrid scale models for fluid flow in cracks and faults are needed for large scale calculations.
- They can be carried out within conventional interface elements, within an extended finite element framework as well as using isogeometric finite element analysis.